CYLINDRICAL WAVES PROPAGATING ACROSS A MAGNETIC FIELD IN A RAREFIED PLASMA

Yu. A. Berezin

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 107–110, 1966

It is well known that the profile of finite-amplitude waves in a rarefied plasma (where the mean free path of the particles is considerably greater than the other characteristic dimensions) is determined by two competing processes: nonlinear twisting and "smearing" as a result of dispersion effects [1, 2]. For waves propagating across a magnetic field in a cold rarefied plasma the dispersion law is such that the phase velocity of small oscillations of the type under consideration falls off as the wavelength decreases (negative dispersion). Such a dispersion law results in the possible existence of stationary "compression" waves of finite amplitude (isolated and periodic), which have been studied fairly fully. A series of papers [3–5] have dealt with nonstationary plane waves moving across a magnetic field and excited by raising the magnetic pressure at the plasma-vacuum boundary.

In what follows cylindrical waves propagating in a cold rarefied plasma across a strong magnetic field are investigated by numerical integration of the appropriate system of equations. The results are of significance for experiments in the rapid compression of plasma columns by a magnetic field under conditions when the plasma may be regarded as sufficiently rarefied [6].

1. Basic system of equations. We shall consider motions whose characteristic frequency is considerably less than the electron Larmor frequency

$$\omega_{He} = eH / m_e c, \quad \omega_{He} \ll \omega_{0e} = \sqrt{4\pi N e^2 / m_e}\ .$$

Thus, the plasma may be taken to be quasi-neutral, $N_i = N_e = N$. We shall also neglect the gas-kinetic pressure in comparison with the magnetic pressure ($p \ll H^2/8\pi$) since the plasma is assumed to be cold. As a result of this the motion of the electrons and ions is determined basically by the self-consistent electromagnetic field.

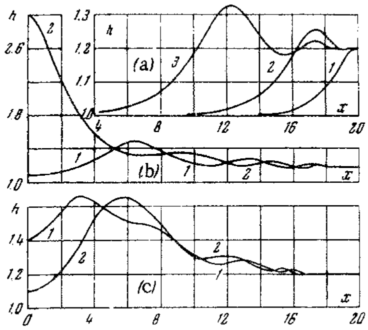

Fig. 1

Under the conditions indicated, the equations of macroscopic motion of the electron and ion components and Maxwell's equations have the following form:

$$m_i \frac{d\mathbf{v}_i}{dt} = e\mathbf{E} + \frac{e}{c}[\mathbf{v}_i \mathbf{H}] + \nu m_e (\mathbf{v}_e - \mathbf{v}_i),$$

$$m_e \frac{d\mathbf{v}_e}{dt} = -e\mathbf{E} - \frac{e}{c}[\mathbf{v}_e \mathbf{H}] - \nu m_e (\mathbf{v}_e - \mathbf{v}_i), \tag{1.1}$$

$$\mathrm{rot}\,\mathbf{H} = \frac{4\pi e N}{c}(\mathbf{v}_i - \mathbf{v}_e), \quad \mathrm{rot}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

$$\frac{\partial N}{\partial t} + \mathrm{div}(N\mathbf{v}_i) = 0, \quad \frac{\partial N}{\partial t} + \mathrm{div}(N\mathbf{v}_e) = 0, \quad \mathrm{div}\,\mathbf{H} = 0\ .$$

Here $\mathbf{v}_i (\mathbf{v}_e)$ is the macroscopic velocity of ions (electrons),

$$\frac{d\mathbf{v}_i}{dt} = \frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \nabla)\mathbf{v}_i, \qquad \frac{d\mathbf{v}_e}{dt} = \frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \nabla)\mathbf{v}_e,$$

ν is the ion-electron collision frequency (for greater generality some friction between plasma components has been introduced into the equations of motion).

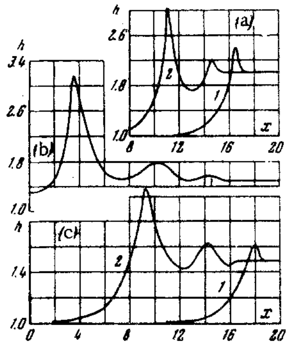

Fig. 2

We introduce the velocity of the center of mass,

$$\mathbf{U} = \frac{m_i \mathbf{v}_i + m_e \mathbf{v}_e}{m_i + m_e}\ . \tag{1.2}$$

Then the electron and ion velocities are written as

$$\mathbf{v}_i = \mathbf{U} + \frac{m_e c}{4\pi e N (m_i + m_e)}\,\mathrm{rot}\,\mathbf{H},$$
$$\mathbf{v}_e = \mathbf{U} - \frac{m_i c}{4\pi e N (m_i + m_e)}\,\mathrm{rot}\,\mathbf{H}\ . \tag{1.3}$$

We now transform the system (1.1) in the following manner. We combine the two first equations of the system and set expressions (1.3) in the resulting expression. We then get an expression for the electric field $\mathbf{E}$ from the first equation of system (1.1) and set it in the induction equation $\partial \mathbf{H}/\partial t = -c\,\mathrm{rot}\,\mathbf{E}$. As a result of these transformations the original system of equations assumes the form

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\nabla)\mathbf{U} = \frac{1}{4\pi N M}[\mathrm{rot}\,\mathbf{H}\cdot\mathbf{H}] - \frac{m_i m_e c^2}{(4\pi e M)^2}\left(\frac{\mathrm{rot}\,\mathbf{H}}{N}\nabla\right)\frac{\mathrm{rot}\,\mathbf{H}}{N}, \qquad \frac{\partial N}{\partial t} + \mathrm{div}(N\mathbf{U}) = 0\ ,$$

$$\frac{\partial \mathbf{H}}{\partial t} = \mathrm{rot}[\mathbf{U}\mathbf{H}] - \frac{c^2}{4\pi}\mathrm{rot}\left(\frac{\mathrm{rot}\,\mathbf{H}}{\sigma}\right) - \frac{m_i c}{4\pi e M}\mathrm{rot}\left[\frac{\mathrm{rot}\,\mathbf{H}}{N}\cdot\mathbf{H}\right] + \left(\frac{m_i c}{4\pi e M}\right)^2 \frac{m_e c}{e}\,\mathrm{rot}\left\{\left(\frac{\mathrm{rot}\,\mathbf{H}}{N}\nabla\right)\frac{\mathrm{rot}\,\mathbf{H}}{N}\right\} -$$
$$- \frac{m_i m_e c^2}{4\pi e^2 M}\,\mathrm{rot}\left\{\frac{\partial}{\partial t}\left(\frac{\mathrm{rot}\,\mathbf{H}}{N}\right) + (\mathbf{U}\nabla)\frac{\mathrm{rot}\,\mathbf{H}}{N} + \left(\frac{\mathrm{rot}\,\mathbf{H}}{N}\nabla\right)\left(\mathbf{U} + \frac{m_e c}{4\pi e M}\frac{\mathrm{rot}\,\mathbf{H}}{N}\right)\right\}, \qquad \mathrm{div}\,\mathbf{H} = 0$$

$$\left(M = m_i + m_e, \quad \sigma = \frac{Ne^2}{m_e \nu}\right). \tag{1.4}$$

Here σ is the conductivity of the plasma. In ordinary magnetohydrodynamics we consider motions whose frequency is much less than the ion-cyclotron frequency $\omega_{Hi} = eH / m_i c$; in addition, the plasma is considered to be fairly dense. The equations corresponding to the approximation of ordinary magnetohydrodynamics may be obtained from system (1.1) by leaving only the term

$$\frac{1}{4\pi NM}(\mathrm{rot}\,\mathbf{H}\times\mathbf{H})$$

on the right side of the first equation, and only the terms

$$\mathrm{rot}\,[\mathbf{U}\times\mathbf{H}], \qquad \frac{c^2}{4\pi}\,\mathrm{rot}\,\frac{\mathrm{rot}\,\mathbf{H}}{\sigma}$$

on the right side of the third equation.

If, as is usual, we consider that σ = const, then the system of equations may in this case be written in the form

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\,\mathbf{U} = \frac{1}{4\pi NM}(\mathrm{rot}\,\mathbf{H}\times\mathbf{H})\;\frac{\partial N}{\partial t} + \mathrm{div}\,(N\cdot\mathbf{U}) = 0,$$

$$\frac{\partial \mathbf{H}}{\partial t} = \mathrm{rot}\,(\mathbf{U}\times\mathbf{H}) + \frac{c^2}{4\pi\sigma}\,\Delta\mathbf{H}\;\mathrm{div}\,\mathbf{H} = 0, \tag{1.5}$$

which coincides with the system of equations of magnetohydrodynamics given, for example, in [7]. The remaining terms in system (1.4) describe dispersion effects characteristic for a rarefied plasma in a strong magnetic field and are connected with taking into account electron inertia as well as plasma gyrotropy. We note that taking these terms into account allows us to treat the region of frequencies

$$\omega \geqslant \omega_{H_i}.$$

Ordinary magnetohydrodynamics "works" in the region of frequencies

$$\omega \ll \omega_{H_i}.$$

On the basis of system (1.4), we shall consider one-dimensional nonstationary cylindrical waves propagating strictly across the magnetic field. The corresponding problem is stated as follows. At the initial moment a homogeneous cold plasma of density N_0 composed of electrons and singly ionized ions fills an infinitely long cylinder of radius a, and there is a uniform magnetic field H_0 along the axis of the cylinder (z axis). The magnetic field at the plasma-vacuum boundary subsequently begins to increase according to some specific law. As a result of the increasing pressure at the boundary a magnetic disturbance propagates towards the axis of the cylinder and the plasma column begins to be compressed.

In the cylindrical waves under consideration here, only the magnetic field component H_z directed along the initial field H_0, and the velocity component U_r directed perpendicular to the wave front differ from zero.

Then system (1.4) reduces to the form

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial r} = -\frac{1}{4\pi NM}H\frac{\partial H}{\partial r} + \frac{m_i m_e c^2}{(4\pi eM)^2}\frac{1}{r}\left(\frac{1}{N}\frac{\partial H}{\partial r}\right)^2,$$

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial r}(NU) + \frac{NU}{r} = 0, \qquad \frac{\partial H}{\partial t} + \frac{\partial}{\partial r}(UH) + \frac{UH}{r} =$$

$$= \frac{\partial}{\partial r}\left(\frac{c^2}{4\pi\sigma}\frac{\partial H}{\partial r}\right) + \frac{c^2}{4\pi\sigma}\frac{1}{r}\frac{\partial H}{\partial r} + \frac{m_i m_e c^2}{4\pi e^2 M}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial t} + \right.$$

$$\left. + U\frac{\partial}{\partial r}\right)\left(\frac{r}{N}\frac{\partial H}{\partial r}\right) \quad (H = H_z,\ U = U_r). \tag{1.6}$$

The dispersion effects are described by the last term on the right-hand side of the induction equation and are caused by the electron intertia.

In order to solve system (1.6) it is convenient to pass to the Lagrangian coordinates r_0, t ($N_0 r_0 dr_0$ = Nrdr, N_0 is the density at the initial moment, r_0 is the initial coordinate of a plasma particle) and to the dimensionless variables

$$h = \frac{H}{H_0}, \quad V = \frac{N_0}{N}, \quad u = \frac{U}{V_A}, \quad x = \frac{r\omega_{0e}}{c},$$

$$\tau = \frac{V_A\omega_{0e}}{c}t \quad (\text{or } \tau = \omega^\circ t), \quad \left(\omega_{0e} = \left(\frac{4\pi N_0 e^2}{m_e}\right)^{1/2},\right.$$

$$\left. V_A = \frac{H_0}{\sqrt{4\pi N_0 M}},\ \omega^\circ = \frac{eH_0}{\sqrt{m_i m_e}c}\right). \tag{1.7}$$

Setting (1.7) in (1.6), we obtain

$$\frac{\partial V}{\partial \tau} = \frac{1}{x_0}\frac{\partial}{\partial x_0}(xu), \qquad \frac{\partial u}{\partial \tau} = -\frac{1}{2}\frac{x}{x_0}\frac{\partial}{\partial x_0}(h^2) + \frac{1}{x}\left(\frac{x}{x_0}\frac{\partial h}{\partial x_0}\right)^2,$$

$$\frac{\partial}{\partial \tau}\left\{Vh - \frac{1}{x_0}\frac{\partial}{\partial x_0}\left(\frac{x^2}{x_0}\frac{\partial h}{\partial x_0}\right)\right\} = \frac{\nu}{\omega^\circ}\frac{1}{x_0}\frac{\partial}{\partial x_0}\left(\frac{x^2}{x_0}\frac{\partial h}{\partial x_0}\right). \tag{1.8}$$

In the case of infinite conductivity the last equation of system (1.8) may be integrated once more to give

$$Vh = 1 + \frac{1}{x_0}\frac{\partial}{\partial x_0}\left(\frac{x^2}{x_0}\frac{\partial h}{\partial x_0}\right). \tag{1.9}$$

This equation may be called the equation of state with a differential coupling, since it gives the relation between the plasma density V^{-1} and the magnetic field h (and consequently the magnetic pressure).

2. Results and discussion. The system of equations (1.8) was solved on an electronic computer by reducing it to finite-difference equations for the following initial and boundary conditions:

$$u(x_0, 0) = 0, \quad V(x_0, 0) = h(x_0, 0) = 1,$$

$$u(0, \tau) = 0, \qquad \frac{\partial h}{\partial x_0}(0, \tau) = 0, \tag{2.1}$$

$$h(a^\circ, \tau) = 1 + A(1 - e^{-\omega\tau}),$$

$$(a^\circ = a\omega_{0e} / c). \tag{2.2}$$

Here a° is the dimensionless radius of the plasma column.

Figure 1a gives the profile of the magnetic field as a function of the Euler coordinate x at different moments of time for the most characteristic stages of the process under consideration (where curves 1, 2, 3 correspond to values $\tau \approx 2, 5, 10$). A magnetic field of comparatively small amplitude with $h^\circ = 1 + 0.2\,(1 - e^{-10\tau})$ is given on the moving (in Euler coordinates) surface of the plasma column. The initial radius of the cylinder is chosen to be 20 in units of c/ω_{0e}. In addition, dissipation is neglected (i.e., the calculation is performed for the condition $\nu = 0$).

For times which are small the profile of the magnetic field does not differ from the profile obtained in ordinary magnetohydrodynamics. The disturbance begins to propagate from the boundary of the column towards the axis with a velocity roughly equal to the Alfven velocity $H / \sqrt{4\pi NM}$. Since the magnetic field pressure at the boundary is comparatively small, this boundary moves slowly and the column contracts. At subsequent moments in time the wave profile begins to become steeper, since the parts with a stronger magnetic field propagate faster than the parts with a weaker magnetic field. In ordinary magnetohydrohynamics a discontinuity is then formed—a shock wave. In our case when the dimension of the region in which a noticeable change in flow parameters occurs becomes equal in order of magnitude to the dispersion length c/ω_{0e}, dispersion effects begin to play a part and under the conditions specific to the problem tend to compensate the nonlinear twisting of the wave profile, and a smooth flow without discontinuities results. The wave profile acquires an oscillatory character: in accordance with the negative dispersion law "peaks" and "valleys" appear in the magnetic field behind the leading front, which gradually transfer the field values from the level maintained at the boundary to the undisturbed value. The linear width of these magnetic field "peaks," which may be interpreted as a train of isolated waves of increasing amplitude, is roughly equal to 2 c/ω_{0e}. As the wave front moves towards the axis, its velocity increases as a result of the increase of the magnetic field. After a time ≈19 the leading front reaches the axis of the column. At this moment a build-up of the magnetic field occurs, the field at the axis reaches a value h ≈ 3, and the density of the plasma increases threefold (Fig. 1b, where curves 1, 2 correspond to values τ = 15, 19). Subsequently an essentially nonstationary reflection of cylindrical waves from the axis takes place; as can be seen from Fig. 1c (curves 1, 2 correspond to the values τ = 21, 22), the maximum of the magnetic field moves rapidly from the axis in the direction of the boundary of the plasma column, and the field at the axis decreases. At subsequent moments of time an interaction occurs between the reflected "part" of the disturbance and waves which continue to come from the boundary, since some constant level of the magnetic field is maintained at the boundary, and the picture becomes difficult to interpret.

Figure 2c gives the magnetic field profile for the cases when the amplitude of the magnetic field at the boundary has a large value, namely, $A = 0.5$ (Fig. 2c, where the curves 1, 2 correspond to values $\tau = 3, 9.5$), and also in Fig. 2b, where the curve is for $\tau = 13$ and $A = 1$ (Fig. 2a, where the curves 1, 2 correspond to the values $\tau = 3, 6$). Naturally, the velocity of the propagating wave becomes greater and the maxima of the magnetic field oscillations also increase. In the case $A = 0.5$ a cylindrical wave covers a distance $x = 16$ in a time ~ 13, and the magnetic field strength increases to $h \approx 3$. The specific volume V here tends to zero (the density increases sharply), and the values of the Euler coordinates corresponding to neighboring "Lagrangian points" become almost coincident. This indicates that for $h \gtrsim 3$ an "intersection of particle trajectories" occurs, since for such small magnetic field amplitudes dispersion effects cannot compensate for the nonlinear twisting; thus the wave apparently "breaks" and forms a region of multi-stream motion [1]. The further course of the process can no longer be followed with the help of Eqs. (1.6) or (1.8), which do not describe multi-stream flow, and another model must be constructed. Such a situation occurs for the case when $A = 1$ (Fig. 2a), when the magnetic field increases to three times the value of the undisturbed field in a time $\tau = 6$. The question of calculation of the flow in a rarefied plasma after the "breaking" of the wave (when the amplitude of the magnetic field in the wave becomes greater than $3H_0$) still remains open, since the method of artificial viscosity usually employed in hydrodynamics for calculating discontinuities is inapplicable here, because in the given situation it does not correspond to the facts.

Calculations were also carried out at a finite value of the conductivity $\nu \neq 0$, $\sigma \neq \infty$, which correspond to the introduction of small dissipative terms into the equations. These calculations show that the character of cylindrical wave propagation and the propagation of finite-amplitude plane waves remains basically the same as in the case of an ideal plasma (for $\sigma = \infty$). The only difference is that the amplitude of "compression" waves is somewhat decreased and the profile is somewhat "smeared out."

The author is grateful to R. Z. Sagdeev and N. N. Yanenko for valuable advice and discussion, and also to G. A. Maksima and E. A. Tsvetov for their assistance in the work.

REFERENCES

1. A. A. Vedenov, E. P. Velikhov and R. Z. Sagdeev, "Nonlinear oscillations of a rarefied plasma," in: Nuclear Syntheses [in Russian], vol. 1, no. 1, 1961.

2. R. Z. Sagdeev, "Collective processes and shock waves in a rarefied plasma," in: Problems of Plasma Theory [in Russian], Atomizdat, 1964.

3. J. Adlam and J. Allen, "Collision-free hydromagnetic disturbances of large amplitude in a plasma," Proc. Phys. Soc., vol. 75, no. 485, 1960.

4. P. L. Auer, H. Hurwitz, and R. W. Kilb, "Low Mach number magnetic compression waves in a collision-free plasma," Phys. Fluids, vol. 4, no. 9, 1961.

5. K. W. Morton, "Finite-amplitude compression waves in a collision-free plasma," Phys. Fluids, vol. 7, no. 11, 1964.

6. A. M. Iskol'dskii, R. Kh. Kurtmullaev, Yu. E. Nesterikhin, and A. G. Ponomarenko, "Experiments with a collisionless shock wave in a plasma," ZhETF, vol. 46, no. 8, 1964.

7. L. D. Landau and E. M. Livshitz, Electrodynamics of Continua [In Russian], Gostekhizdat, 1957.

22 March 1965 Novosibirsk